Pomeau-Manneville maps are global-local mixing

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Abstract

We prove that a large class of expanding maps of the unit interval with a C^2-regular indifferent fixed point in 0 and full increasing branches are global-local mixing. This class includes the standard Pomeau-Manneville maps $T(x) = x + x^{p+1} \bmod 1$ ($p \geq 1$), the Liverani-Saussol-Vaienti maps (with index $p \geq 1$) and many generalizations thereof.

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1 Introduction

By 'Pomeau-Manneville map' one generally means a piecewise expanding map of $[0, 1]$ with two increasing surjective branches and an indifferent fixed point in 0. These maps are so named after Pomeau and Manneville who, in the late 1970s, studied numerically approximated versions of them to investigate the phenomenon of intermittency in physics [11, 10].

The first examples of this kind were the maps $T_{PM}(x) = x + x^{p+1} \bmod 1$, with $p \in \mathbb{R}^+$. Throughout this paper we refer to them as *classical Pomeau-Manneville*

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(PM) maps. In 1999, Liverani, Saussol and Vaienti [9] introduced a somewhat simpler one-parameter family:

$$T_{LSV}(x) = \begin{cases} x + 2^p x^{p+1}, & 0 \le x \le \frac{1}{2}; \\ 2x - 1, & \frac{1}{2} < x \le 1, \end{cases} \tag{1}$$

where again $p \in \mathbb{R}^+$ (although in [9] only the case $0 < p < 1$ was treated). These are nowadays called *LSV maps.* Each of these dynamical systems has long been known to possess an absolutely continuous invariant measure μ, unique up to factors, whose density behaves like x^{-p}, as $x \to 0^+$ [12]. Thus μ is infinite if and only if $p \ge 1$. For such choice of p, both the classical PM and the LSV maps have become paradigms not just of non-uniformly expanding/hyperbolic maps, but also of dynamical systems preserving an infinite measure. They are undoubtedly the most common examples in the field of infinite ergodic theory.

In this paper we prove that a large class of piecewise expanding maps of the unit interval with an indifferent fixed point in 0 — including the above ones and many more — are global-local mixing. This means that for any $F \in L^\infty(\mu)$ which admits a finite *infinite-volume average*

$$\overline{\mu}(F) := \lim_{a \to 0^+} \frac{1}{\mu([a,1])} \int_a^1 F \, d\mu, \tag{2}$$

and any $g \in L^1(\mu)$, one has

$$\lim_{n \to \infty} \int_0^1 (F \circ T^n) g \, d\mu = \overline{\mu}(F) \int_0^1 g \, d\mu. \tag{3}$$

(In truth, we have a slightly weaker result in the case where the *index* of the map p equals 1, see below.) We call all functions like F 'global observables' and all functions like g 'local observables'. We indicate them, respectively, with uppercase and lowercase letters.

The notion of global-local mixing has received increasing attention lately in infinite ergodic theory; cf. [5, 7, 1, 2, 3, 4] and references therein. An example of its usefulness is the fact that, for maps with indifferent fixed points, it provides interesting unconventional limit theorems, cf. [1, Sect. 3] and [2, Sect. 3].

The class of systems that we consider here is given by piecewise expanding maps $T : (0,1] \longrightarrow (0,1]$ with a finite or countable number of increasing surjective branches and such that, for $x \to 0^+$, $T(x) = x + \kappa x^{p+1} + o(x^{p+1})$, with $\kappa > 0$ and $p \ge 1$. We also assume a certain condition on the growth of the branches, see (A5) below, together with standard hypotheses (regularity, distortion bounds, etc.). The class includes:

- *generalized classical PM maps*, that is, maps of the form $T(x) = x + \kappa x^{p+1} \mod 1$, with $\kappa \in \mathbb{Z}^+$;

- *generalized LSV maps*, that is, maps whose branch at 0 is given by $T|_{(0,a_1]}(x) = x + \kappa x^{p+1}$, where now $\kappa \in \mathbb{R}^+$, and the other branches are piecewise linear and increasing;

- suitable perturbations of the above types.

In [2, Rmks. 2.15-2.16] the present authors and P. Giulietti had already discussed global-local mixing for Pomeau-Manneville maps, albeit not in particularly decisive terms. The problem there was that the proof of global-local mixing required a rather precise knowledge of $h_\mu = \frac{d\mu}{dm}$, the (infinite) invariant density, which is *not* known in general. One of the few cases in which it is known is that of the map $T(x) = x + x^2$ mod 1, which was therefore proved to be global-local mixing in [2, Rmk. 2.15]. No other case of classical PM or LSV map was covered so far.

The techniques used in this work are substantial refinements of those employed in [2], in that they only require to know the behavior of the singularity of h_μ around 0. But a classical result of Thaler (see Theorem 2.1*(a)* below) provides exactly that, for an ample class of maps, as a function of the index p. An intermediate step for our main result (Theorem 2.4) is the proof of global-local mixing relative to a measure ν_p whose singular exponent at 0 is exactly one unit less than that of μ. What is remarkable is that global-local mixing relative to ν_p is completely equivalent to global-local mixing relative to μ, for all $p > 1$. For $p = 1$, the two results are *almost* equivalent, in that the class of global observables is slightly smaller than the optimal class, cf. Theorem 2.4*(c)*.

The paper is organized as follows. In Section 2 we lay out the necessary mathematical background and state our results; then we present a list of examples to which our main theorem applies. In Section 3 we prove global-local mixing for a large class of maps $\mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ with an "indifferent fixed point at ∞". Aside from its own worth, this result is instrumental in the proof of the main theorem, which is given in Section 4. Finally, certain technical results are proved in the Appendix.

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2 Setup and results

We now give a precise definition of the class of maps we consider in this article. A finite or infinite sequence of numbers $0 = a_0 < a_1 < \ldots < a_k < \ldots \leq 1$ is given. If the sequence is finite, its last element is denoted a_N and it equals 1; in this case we set $\mathcal{J} := \{0, \ldots, N-1\}$. If the sequence is infinite, $\lim_n a_n = 1$; in this case we set

$\mathcal{J} := \mathbb{N}$. For $j \in \mathcal{J}$, denote $I_j := (a_j, a_{j+1}]$. If m denotes the Lebesgue measure, then $\{I_j\}_{j\in\mathcal{J}}$ is a partition of $(0,1]$ mod m, which acts as the Markov partition of a map $T : (0,1] \longrightarrow (0,1]$ satisfying the following conditions:

(A1) $T|_{I_j}$ possesses a continuous extension $\tau_j : [a_j, a_{j+1}] \longrightarrow [0,1]$ which is strictly increasing, bijective and twice differentiable on (a_j, a_{j+1}).

(A2) There exist $\kappa > 0$, $p \geq 1$ and $b_o \in (0, a_1)$ such that $\tau_0(x) = x + \kappa x^{p+1} + o(x^{p+1})$, as $x \to 0^+$, and τ_0 is strictly convex in $[0, b_o]$. This implies in particular that $\tau_0'(0) = 1$ and $\tau_0'(x) > 1$, for $x \in (0, b_o)$.

(A3) There exists $\Lambda > 1$ such that $T'(x) \geq \Lambda$, for all $x \in [b_o, 1) \setminus \{a_j\}_{j\geq 1}$.

(A4) There exists $K > 0$ such that $\dfrac{|T''(x)|}{(T'(x))^2} \leq K$, for all $x \in (0,1) \setminus \{a_j\}_{j\geq 1}$.

(A5) Set $\phi_j := \tau_j^{-1}$. For all $j \in \mathcal{J}$, $\displaystyle\sum_{k\geq j} \left(\frac{\xi}{\phi_k(\xi)} \right)^{p+1} \phi_k'(\xi)$ is a (not necessarily strictly) increasing function of $\xi \in (0,1)$.

We will refer to τ_j as a *branch* of T and to ϕ_j as the corresponding *inverse branch*. Under (much) more general conditions than the above, Thaler [12, 13, 14] proved the following results.

Theorem 2.1 *Under the assumptions* (A1)-(A4)*:*

(a) T preserves an infinite invariant measure μ which is absolutely continuous w.r.t. the Lebesgue measure m, with (infinite) density

$$h_\mu(x) := \frac{d\mu}{dm}(x) = \frac{H_\mu(x)}{x^p},$$

where H_μ is positive and continuous on $[0,1]$.

(b) Up to multiplicative constants, μ is the unique m-absolutely continuous invariant measure.

(c) T is conservative and exact (w.r.t. m or μ, which is the same).

We now introduce two types of observables on the space $(0,1]$. Let ν be an infinite measure on $(0,1]$ such that $\nu([a,1]) < \infty$, for all $a \in (0,1]$. In particular, ν is σ-finite. In what follows we refer to any such ν as a measure which is *infinite at 0.* Define

$$\mathcal{G}((0,1],\nu) := \left\{ F \in L^\infty((0,1],\nu) \;\middle|\; \exists\, \overline{\nu}(F) := \lim_{a\to 0^+} \frac{1}{\nu([a,1])} \int_a^1 F \, d\nu \right\}. \tag{4}$$

We call any F as in the above definition a *global observable* and say that $\overline{\nu}(F)$ is the *infinite-volume average* of F relative to ν. We also call any $f \in L^1((0,1],\nu)$ a *local observable*. For added clarity, we denote global, respectively local, observables with uppercase, respectively lowercase, letters. In the remainder we use the conventional abbreviation $\nu(f) := \int_0^1 f \, d\nu$.

Definition 2.2 *Given a measure ν which is infinite at 0 and two (sub)classes of global and local observables, respectively $\mathcal{G} \subseteq \mathcal{G}((0,1],\nu)$ and $\mathcal{L} \subseteq L^1((0,1],\nu)$, we say that the map T is* ***global-local mixing*** *w.r.t. $\nu, \mathcal{G}, \mathcal{L}$ if, for all $F \in \mathcal{G}$ and $g \in \mathcal{L}$,*

$$\lim_{n\to\infty} \nu((F \circ T^n) g) = \overline{\nu}(F)\nu(g).$$

If the above can be proved for $\mathcal{G} = \mathcal{G}((0,1],\nu)$ and $\mathcal{L} = L^1((0,1],\nu)$, we say that T is ***fully global-local mixing*** *w.r.t. ν.*

Remark 1 In the context of *infinite-volume mixing* [5, 6], there are a few variants of the definition of global-local mixing. The one we consider here is the most natural among them and is otherwise denoted **(GLM2)**. More importantly, in that framework, the class of global observables $\mathcal{G}((0,1],\nu)$ corresponds to the *exhaustive family* $\mathscr{V} := \{[a,1] \mid 0 < a < 1\}$. An exhaustive family is a collection of finite-measure “large boxes” that are used to define the infinite-volume limit. In the present case, where the reference space $(0,1]$ is one-dimensional and possesses only one “point at infinity” (w.r.t. ν), $\mathscr{V}$ is essentially the smallest choice for the exhaustive family, making $\mathcal{G}((0,1],\nu)$ the largest reasonable class of global observables, relative to ν. Since $L^1((0,1],\nu)$ is the largest class of local observables for which the definition of global-local mixing makes sense, this explains the expression ‘full global-local mixing’.

Definition 2.3 *We say that two measures ν_1 and ν_2 (that are infinite at 0) give rise to two* ***identical*** *infinite-volume averages $\overline{\nu}_1$ and $\overline{\nu}_2$ if $\mathcal{G}((0,1],\nu_1) = \mathcal{G}((0,1],\nu_2)$ and $\overline{\nu}_1(F) = \overline{\nu}_2(F)$ for every $F \in \mathcal{G}((0,1],\nu_1)$. In this case, we write $\overline{\nu}_1 = \overline{\nu}_2$.*

Remark 2 It is obvious that $\overline{\nu}$ only depends on the behavior of ν around zero, in that, for all $F \in \mathcal{G}((0,1],\nu)$ and $0 < b < 1$, $\overline{\nu}(F) = \overline{\nu}(F 1_{(0,b]})$. In particular, if ν_1 is absolutely continuous w.r.t. ν_2 in a neighborhood of 0 and $\frac{d\nu_1}{d\nu_2}(x)$ tends to a positive constant as $x \to 0^+$, then $\overline{\nu}_1 = \overline{\nu}_2$.

Definition 2.2 of global-local mixing has a stronger significance, in the sense of the decorrelation between a global and a local observable, when $\overline{\nu}$ is an invariant functional, i.e., $\overline{\nu}(F \circ T^n) = \overline{\nu}(F)$, for all $n \in \mathbb{N}$. This is the case, for instance, where $\overline{\nu} = \overline{\mu}$ and μ is the invariant measure guaranteed by Theorem 2.1 [2, Prop. 2.3]. Of course, we are especially interested in this case. Nevertheless, in order to state our main result, Theorem 2.4 below, we introduce a special measure which will play a role both in the statement and even more in the proof of the theorem. For $p \geq 1$, let ν_p be the Lebesgue-absolutely continuous measure defined by

$$h_{\nu_p}(x) := \frac{d\nu_p}{dm}(x) = \frac{1}{x^{p+1}} \tag{5}$$

Clearly, ν_p is infinite at 0.

Theorem 2.4 *Let T satisfy* (A1)-(A5). *Then:*

(a) T is fully global-local mixing relative to ν_p.

As concerns the invariant measure μ:

(b) If $p > 1$, T is fully global-local mixing relative to μ as well.

(c) If $p = 1$, then $\mathcal{G}((0,1],\nu_1) \subsetneq \mathcal{G}((0,1],\mu)$, with $\overline{\mu}(F) = \overline{\nu}_1(F)$ for all $F \in \mathcal{G}((0,1],\nu_1)$. Furthermore, T is global-local mixing w.r.t. μ, $\mathcal{G}((0,1],\nu_1)$ and $L^1((0,1],\mu)$.

To be crystal-clear, the second claim of *(c)* states that, for all $F \in \mathcal{G}((0,1],\nu_1)$ and $g \in L^1((0,1],\mu)$,

$$\lim_{n\to\infty} \mu((F \circ T^n)g) = \overline{\mu}(F)\mu(g). \tag{6}$$

Remark 3 Of the five assumptions (A1)-(A5), the last one is certainly the hardest to check. A stronger but more readable assumption is:

(A5)' For all $j \in \mathcal{J}$, $\left(\frac{\tau_j(x)}{x}\right)^{p+1} \frac{1}{\tau_j'(x)}$ is a (not necessarily strictly) increasing function of $x \in (a_j, a_{j+1})$.

In fact, if (A5)' holds, for all j we can compose the function $(\tau_j(x)/x)^{p+1}/\tau_j'(x)$ with $x = \phi_j(\xi)$, which is an increasing function by (A2). Hence $(\xi/\phi_j(\xi))^{p+1}\phi_j'(\xi)$ is increasing in ξ and implies (A5).

2.1 Pomeau-Manneville maps and other examples

We prove Theorem 2.4 in Section 4. For the moment we verify that several classes of well-known intermittent maps of the unit interval satisfy its hypotheses.

1. Generalized classical Pomeau-Manneville (PM) maps. These are maps of the form

$$T(x) = x + \kappa x^{p+1} \mod 1, \tag{7}$$

where $p \geq 1$ and $\kappa \in \mathbb{Z}^+$. The term 'generalized' refers to the fact that the number $N = \kappa + 1$ of branches may be larger than one. Assumptions (A1)-(A4) are easily verified.

We proceed to verify (A5)'. For all $0 \le j \le N-1$, a_j is the unique x such that $x + \kappa x^{p+1} = j$ and $\tau_j(x) = x + \kappa x^{p+1} - j$, whence

$$\left(\frac{\tau_j(x)}{x} \right)^{p+1} \frac{1}{\tau_j'(x)} = \frac{(1 + \kappa x^p - j x^{-1})^{p+1}}{1 + \kappa(p+1)x^p}. \tag{8}$$

Verifying (A5)' is tantamount to verifying that the logarithmic derivative of the above, w.r.t. the variable $z := \kappa x^p$, is non-negative. But

$$\begin{aligned} &\frac{d}{dz} \log \left(\frac{\left(1 + z - j\kappa^{1/p} z^{-1/p}\right)^{p+1}}{1 + (p+1)z} \right) \\ &\qquad = (p+1) \frac{1 + j\kappa^{1/p}(1/p) z^{-1-1/p}}{1 + z - j\kappa^{1/p} z^{-1/p}} - \frac{p+1}{1 + (p+1)z} \\ &\qquad = (p+1) \frac{pz + j\kappa^{1/p} z^{-1/p}(z^{-1} + 2p + 1)/p}{(1 + z - j\kappa^{1/p} z^{-1/p})\,(1 + (p+1)z)} > 0, \end{aligned} \tag{9}$$

for all z such that $1 + z - j\kappa^{1/p} z^{-1/p} > 0$, corresponding to $x > a_j$.

2. Generalized Liverani-Saussol-Vaienti (LSV) maps. These are maps which can have a finite or infinite number of branches. The first branch is

$$\tau_0(x) = x + \kappa x^{p+1}, \tag{10}$$

with $p \ge 1$. Here $\kappa \in \mathbb{R}^+$ with the only obvious constraint that $a_1 + \kappa a_1^{p+1} = 1$, for some $0 < a_1 < 1$. The other endpoints a_j can be fixed freely and, for $j \ge 1$,

$$\tau_j(x) = \frac{x - a_j}{a_{j+1} - a_j}, \tag{11}$$

meaning that τ_j is linear, increasing and surjective. (The actual LSV maps, cf. (1), have $N = 2$ branches and are defined by $\kappa = 2^p$.)

Again, (A1)-(A4) are easily verified, so we check (A5)'. It is convenient to split (A5)' in a number of assumptions, for $j \in \mathcal{J}$:

(A5)$'_j$ $\quad \left(\dfrac{\tau_j(x)}{x} \right)^{p+1} \dfrac{1}{\tau_j'(x)}$ is an increasing function of $x \in (a_j, a_{j+1})$.

We observe that (A5)$'_0$ was already proved in (8)-(9), which in no way depended on the fact that κ was an integer there. Also, (A5)$'_j$ for $j \ge 1$ is immediate since

$$\frac{\tau_j(x)}{x} = \frac{1 - a_j x^{-1}}{a_{j+1} - a_j} \tag{12}$$

is increasing in x and τ_j' is a positive constant.

3. Perturbations of generalized classical PM maps. With reference to the maps of type 1 above, let T be defined by the branches

$$\tau_j(x) = x + \kappa x^{p+1} - j + \eta_j(x), \tag{13}$$

for $j = 0, \dots, N-1$. Here $\eta_j : [a_j, a_{j+1}] \longrightarrow \mathbb{R}_0^+$ is twice differentiable, $p \geq 1$ as usual and $\kappa \in \mathbb{R}^+$. We assume that (A1)-(A4) hold. We now prove that, for a sufficiently small $\varepsilon > 0$, if

$$|\eta_0(x)| \leq \varepsilon x^{2p+1}; \qquad |\eta_0'(x)| \leq \varepsilon x^{2p}; \qquad |\eta_0''(x)| \leq \varepsilon x^{2p-1}, \tag{14}$$

and, for $j \geq 1$,

$$|\eta_j(x)| \leq \varepsilon x^{p+1}; \qquad |\eta_j'(x)| \leq \varepsilon x^{p}; \qquad |\eta_j''(x)| \leq \varepsilon x^{p-1}, \tag{15}$$

T satisfies (A5)' as well.

Using again the convenient variable $z = \kappa x^p \in [0, \kappa]$, we rewrite

$$\left(\frac{\tau_j(x)}{x} \right)^{p+1} \frac{1}{\tau_j'(x)} = \frac{\left(1 + z - j\kappa^{1/p} z^{-1/p} + u_j(z)\right)^{p+1}}{1 + (p+1)(z + v_j(z))}, \tag{16}$$

where $u_j(z) := \kappa^{1/p} z^{-1/p} \eta_j(\kappa^{-1/p} z^{1/p})$ and $v_j(z) := \eta_j'(\kappa^{-1/p} z^{1/p})/(p+1)$ (here η_j' denotes the derivative of η_j w.r.t. x). In other words, $\eta_j(x)/x = u_j(z)$ and $\eta_j'(x) = (p+1)v_j(z)$. The logarithmic derivative of (16) w.r.t. z equals

$$\begin{aligned} &(p+1)\frac{1 + j\kappa^{1/p}(1/p)z^{-1-1/p} + u_j'(z)}{1 + z - j\kappa^{1/p} z^{-1/p} + u_j(z)} - \frac{(p+1)(1 + v_j'(z))}{1 + (p+1)(z + v_j(z))} \\ &=: (p+1)\left(\frac{A(z)}{B(z)} - \frac{C(z)}{D(z)} \right) = \frac{p+1}{B(z)D(z)} \left(A(z)D(z) - B(z)C(z) \right), \end{aligned} \tag{17}$$

where u_j' and v_j' now denote derivatives w.r.t. z. Observe that, for the values of z that we are considering, that is, $\kappa a_j^p < z < \kappa a_{j+1}^p$, both $B(z)$ and $D(z)$ are positive, as they correspond, respectively, to $\tau_j(x)/x$ and $\tau_j'(x)$, with $a_j < x < a_{j+1}$. Proving that $A(z)D(z) - B(z)C(z) \geq 0$, for the mentioned values of z, will thus establish $(\mathrm{A5})_j'$. After some computations and regrouping of similar terms, one has:

$$\begin{aligned} &A(z)D(z) - B(z)C(z) \\ &\quad = \left[u_j'(z) + (p+1)v_j(z) - u_j(z) - v_j'(z) + (p+1)u_j'(z)v_j(z) - u_j(z)v_j'(z) \right] \\ &\qquad + z\left[p + (p+1)u_j'(z) - v_j'(z) \right] \\ &\qquad + z^{-1/p} j\kappa^{1/p} \left[2 + 1/p + v_j'(z) \right] \\ &\qquad + z^{-1-1/p} j\kappa^{1/p} \left[1/p + (1 + 1/p)v_j(z) \right]. \end{aligned} \tag{18}$$

Let us verify $(\mathrm{A5})_0'$. In terms of u_0 and v_0, the conditions (14) read:

$$|u_0(z)| \leq \varepsilon_o z^2; \quad |u_0'(z)| \leq \varepsilon_o z; \quad |v_0(z)| \leq \varepsilon_o z^2, \quad |v_0'(z)| \leq \varepsilon_o z, \tag{19}$$

for some $\varepsilon_o > 0$ proportional to ε. Here we have used that $\frac{dx}{dz} = \frac{1}{p}\kappa^{1/p} z^{(1-p)/p}$. So, for ε (and thus ε_o) small enough, the contribution of all the terms in the second and third lines of (18) can be made bigger than, say, $(p/2)z > 0$, for all $z \in (0, \kappa\, a_1^p)$. Since the fourth and fifth lines of (18) are null for $j = 0$, this gives $(A5)'_0$.

In the case $j \geq 1$, the conditions (15) imply that

$$|u_j(z)| \leq \varepsilon_o z; \quad |u_j'(z)| \leq \varepsilon_o; \quad |v_j(z)| \leq \varepsilon_o z, \quad |v_j'(z)| \leq \varepsilon_o, \tag{20}$$

for a small ε_o. In this case we must consider values of $z \in (\kappa a_j^p, \kappa a_{j+1}^p)$. Since in this case $pz > p\kappa a_j^p > 0$, the above hypotheses are enough to ensure that the second and third lines of (18) can be made bigger than a positive constant. By the same reasoning, and the fact that z is bounded above, we can say the same about the fourth and fifth lines. This proves $(A5)'_j$.

4. Perturbations of generalized LSV maps. In view of the maps of type 2 above, consider a T given by

$$\tau_0(x) = x + \kappa x^{p+1} + \eta_0(x); \tag{21}$$

$$\tau_j(x) = \frac{x - a_j}{a_{j+1} - a_j} + \eta_j(x) \qquad (j \geq 1). \tag{22}$$

Once again, we assume (A1)-(A4). (In particular, we have $\eta_j(a_j) = \eta_j(a_{j+1}) = 0$.) In addition, suppose that η_0 verifies conditions (14), with the same ε determined *a fortiori* to work for the example 3, and, for $j \geq 1$,

$$\eta_j''(x) \leq 0; \qquad \eta_j(x) - x\eta_j'(x) \leq \frac{a_j}{a_{j+1} - a_j}. \tag{23}$$

Let us observe that it must be $0 \leq \eta_j(x) \leq 1$, the first inequality coming from the concavity of η_j and the second from (22) and (A1).

The above hypotheses imply (A5)'. In fact, $(A5)'_0$ holds by construction. As for $(A5)'_j$, with $j \geq 1$, observe that $\tau_j'' = \eta_j'' \leq 0$, so that $1/\tau_j'$ is increasing. Furthermore,

$$\frac{d}{dx}\left(\frac{\tau_j(x)}{x}\right) = \frac{a_j}{a_{j+1} - a_j}\,\frac{1}{x^2} + \frac{\eta_j'(x)}{x} - \frac{\eta_j(x)}{x^2} \tag{24}$$

is non-negative if and only if the second inequality of (23) holds.

We conclude this section by emphasizing that in no way are the conditions presented in examples 3 and 4 necessary for (A5)'. They have been chosen only because of their simplicity and capacity to generate many examples. When faced with a specific map T, the most reasonable course of action is to simply check (A1)-(A5) or, if calculating the inverse branches of T is too cumbersome, (A1)-(A4) and (A5)'.

3 Maps on the half-line

In this section we introduce a class of Markov maps of the half-line $\mathbb{R}_0^+ := [0, +\infty)$ that are analogous to the maps of the unit interval discussed earlier. For these maps we prove full global-local mixing w.r.t. a class of measures, including in many cases the Lebesgue measure. This result is interesting in its own right and will be the basis for the proof of Theorem 2.4, which we present in the next section.

A map $T : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ is defined as follows. There exists a finite or infinite sequence of numbers $a_1 > a_2 > \ldots > a_k > \ldots \geq 0$. If the sequence is finite, its last element is $a_N := 0$; in this case we set $\mathcal{J} := \{0, \ldots, N-1\}$. If the sequence is infinite, $\lim_n a_n = 0$; in this case we set $\mathcal{J} := \mathbb{N}$. For $j \in \mathcal{J}$, denote $I_j := [a_{j+1}, a_j)$, where we have conventionally put $a_0 := +\infty$. The following assumptions hold:

(B1) For all $j \in \mathcal{J}$, $\tau_j := T|_{I_j}$ is an increasing diffeomorphism $I_j \longrightarrow \mathbb{R}_0^+$, up to a_{j+1} and 0, on the domain and codomain, respectively.

(B2) T is exact w.r.t. m, the Lebesgue measure on $\mathbb{R}_0^+$.

(B3) Set $\phi_j := \tau_j^{-1}$. For all $j \in \mathcal{J}$, $\displaystyle\sum_{k \geq j} \phi_k'$ is a (not necessarily strictly) decreasing function of $\mathbb{R}_0^+$.

Let ν be an infinite, locally finite, measure on $\mathbb{R}_0^+$. In analogy to (4), we define the class of global observables relative to ν to be

$$\mathcal{G}(\mathbb{R}_0^+, \nu) := \left\{ F \in L^\infty(\mathbb{R}_0^+, \nu) \;\middle|\; \exists\, \overline{\nu}(F) := \lim_{a \to +\infty} \frac{1}{\nu([0,a])} \int_0^a F \, d\nu \right\}. \tag{25}$$

Again we call local observable any $f \in L^1(\mathbb{R}_0^+, \nu)$, and use the abbreviation $\nu(f) = \int_0^\infty f \, d\nu$. In view of Remark 1, we emphasize that the above choice of global observables corresponds to the exhaustive family $\mathscr{V} := \{[0,a] \mid a > 0\}$ for $\mathbb{R}_0^+$, and it is effectively the largest class of global observables relative to ν. The definitions of global-local mixing and identical infinite-volume averages (Definitions 2.2 and 2.3) naturally carry over to this setting.

In this context m denotes the Lebesgue measure on $\mathbb{R}_0^+$. Also, for $0 < q \leq 1$, we introduce the Lebesgue-absolutely continuous measure λ_q defined by the (infinite) density

$$h_{\lambda_q}(y) := \frac{d\lambda_q}{dm}(y) = \frac{1}{(1+y)^q} \tag{26}$$

The rest of the section is devoted to proving the following result:

Theorem 3.1 *Under assumptions* (B1)-(B3)*:*

(a) T is fully global-local mixing relative to m.

(b) T is fully global-local mixing relative to λ_q, for all $q \in (0,1)$.

(c) $\mathcal{G}(\mathbb{R}_0^+, m) \subsetneq \mathcal{G}(\mathbb{R}_0^+, \lambda_1)$, *with* $\overline{\lambda}_1(F) = \overline{m}(F)$ *for all* $F \in \mathcal{G}(\mathbb{R}_0^+, m)$. *Moreover,* T *is global-local mixing w.r.t.* λ_1, $\mathcal{G}(\mathbb{R}_0^+, m)$, $L^1(\mathbb{R}_0^+, \lambda_1)$.

PROOF. Let us start with assertion *(a)*. We use the same technique as in the proof of [2, Thm. 5.2]. We still write all the arguments because the proof in [2] was given for the case where T is a map of the unit interval and the measure used there was invariant, and not just non-singular.

In what follows we write L^1 for $L^1(\mathbb{R}_0^+, m)$. Let $P = P_{T,m} : L^1 \longrightarrow L^1$ be the Perron-Frobenius operator of T, i.e., the transfer operator of T relative to the Lebesgue measure m. It is well known that P acts as

$$Pg(y) = \sum_{j \in \mathcal{J}} \phi_j'(y)\, g(\phi_j(y)). \tag{27}$$

Also, it is a positive operator and, for all $g \geq 0$, $\|Pg\|_1 = \|g\|_1$. Thus, for a general $g \in L^1$, $\|Pg\|_1 \leq \|g\|_1$. In our case P enjoys this crucial property as well:

Lemma 3.2 *If* $g \in L^1$ *is a decreasing (thus necessarily non-negative) function of* $\mathbb{R}_0^+$, *then so is* Pg. *In other words,* P *preserves the cone of the decreasing functions in* L^1.

PROOF OF LEMMA 3.2. The above-mentioned cone is spanned by the finite linear combinations, with positive coefficients, of the functions $1_{[0,a]}$, for $a > 0$. Since P is a contraction operator, namely $\|P\| \leq 1$, it suffices to prove the assertion for $g = 1_{[0,a]}$.

Let $j \in \mathcal{J}$ be the unique index such that $a \in I_j$ and set $b := \tau_j(a)$. By (27)

$$P1_{[0,a]}(y) = \begin{cases} \displaystyle\sum_{k \geq j} \phi_k'(y), & y < b; \\ \displaystyle\sum_{k \geq j+1} \phi_k'(y), & y \geq b. \end{cases} \tag{28}$$

Now, the two branches on the above r.h.s. are decreasing functions of y by (B3) and the gap between them, at $y = b$, is $-\phi_j'(b)$, which is negative by (B1). Q.E.D.

Lemma 3.2 guarantees that every decreasing $g \in L^1$ is a *persistently decreasing* local observable, relative to P. This means that, for all $n \in \mathbb{N}$, $P^n g$ is decreasing. (The notion of *persistently monotonic* observable was introduced in [2].)

We now need the following general lemma:

Lemma 3.3 *Let* T *be a non-singular, exact endomorphism of the* σ*-finite, infinite measure space* $(\mathcal{M}, \mathscr{A}, \nu)$. *Then:*

(a) Given $F \in L^\infty$, *if there exists* $\overline{F} \in \mathbb{R}$ *such that the limit*

$$\lim_{n\to\infty} \nu((F \circ T^n)g) = \overline{F}\nu(g)$$

holds for some $g \in L^1$, *with* $\nu(g) \neq 0$, *then it holds for all* $g \in L^1$.

(b) T is ***local-local mixing****, that is, for all $f \in L^\infty \cap L^1$ and $g \in L^1$,*

$$\lim_{n\to\infty} \nu((f \circ T^n)g) = 0.$$

The above statements were essentially proved in [6, Lem. 3.6 and Thm. 3.5(b)]. However, since they were stated with stronger hypotheses there, and also in the interest of completeness, we present the proof of Lemma 3.3 in the Appendix A.1.

Assumption (B2) and Lemma 3.3*(a)* show that it suffices to verify global-local mixing for a single persistently decreasing local observable g with $\|g\|_1 = 1$. Also, by possibly centering the global observable (i.e., using $F - \overline{m}(F)$ instead of F), we can always assume that $\overline{m}(F) = 0$. Since, by definition of P, $m((F \circ T^n)g) = m(F\, P^n g)$, it remains to verify that

$$\lim_{n\to\infty} m(F\, P^n g) = 0. \tag{29}$$

Fix $\varepsilon > 0$. By definition of $\overline{m}$ we can find $M \in \mathbb{R}^+$ such that, for all $a \geq M$,

$$\frac{1}{a} \left| \int_0^a F(y)\, dy \right| < \frac{\varepsilon}{2}. \tag{30}$$

For $y \in \mathbb{R}_0^+$ and $n \in \mathbb{N}$, set

$$\gamma_n(y) = \gamma_{n,M}(y) := \min\{P^n g(M), P^n g(y)\}. \tag{31}$$

Since g is persistently decreasing, γ_n is a positive, (non strictly) decreasing function, which is constant on $[0, M]$. It is a local observable because $\|\gamma_n\|_1 \leq \|P^n g\|_1 = \|g\|_1 = 1$. We have

$$m(F\, P^n g) = \int_0^\infty F \gamma_n \, dm + \int_0^M F(P^n g - \gamma_n)\, dm =: \mathcal{I}_1 + \mathcal{I}_2. \tag{32}$$

To estimate $\mathcal{I}_2$, let us notice that

$$0 \leq \int_0^M (P^n g - \gamma_n)\, dm \leq \int_0^M P^n g\, dm = m\big((1_{[0,M]} \circ T^n) g\big) \tag{33}$$

As $n \to \infty$, this term vanishes by Lemma 3.3*(b)*. So, for all n large enough,

$$|\mathcal{I}_2| \leq \|F\|_\infty \int_0^M (P^n g - \gamma_n)\, dm \leq \frac{\varepsilon}{2}. \tag{34}$$

Let us now study $\mathcal{I}_1$. We introduce the generalized inverse of the function γ_n, which we define to be $\gamma_n^{-1}(r) := \inf\left\{y \in \mathbb{R}_0^+ \mid \gamma_n(y) \leq r\right\}$. Evidently, γ_n^{-1} is decreasing and $\gamma_n^{-1}(r) = 0$ for $r \geq \gamma_n(M)$. We rewrite $\mathcal{I}_1$ as a double integral:

$$\mathcal{I}_1 = \int_0^\infty F(y) \left(\int_0^{\gamma_n(y)} dr \right) dy = \int_0^{\gamma_n(M)} \left(\int_0^{\gamma_n^{-1}(r)} F(y)\, dy \right) dr, \tag{35}$$

where in the second equality we have used Fubini's Theorem to interchange the order of integration. Therefore, using (30) with $a := \gamma_n^{-1}(r) \geq M$, we obtain

$$\begin{aligned} |\mathcal{I}_1| &\leq \int_0^{\gamma_n(M)} \left| \int_0^{\gamma_n^{-1}(r)} F(y)\, dy \right| dr \\ &\leq \frac{\varepsilon}{2} \int_0^{\gamma_n(M)} \int_0^{\gamma_n^{-1}(r)} dy\ dr \\ &= \frac{\varepsilon}{2} \int_0^{\infty} \int_0^{\gamma_n(y)} dr\ dy \\ &= \frac{\varepsilon}{2} m(\gamma_n) \leq \frac{\varepsilon}{2}. \end{aligned} \tag{36}$$

Observe that this estimate holds for all n. Together with (34) and (32), it proves (29) and thus assertion *(a)* of the theorem.

The other assertions will follow easily from the following lemma, which is proved in Appendix A.2.

Lemma 3.4 *In view of definitions (25)-(26):*

(a) For $q \in (0,1)$, $\mathcal{G}(\mathbb{R}_0^+, \lambda_q) = \mathcal{G}(\mathbb{R}_0^+, m)$ and $\overline{\lambda}_q(F) = \overline{m}(F)$, for all $F \in \mathcal{G}(\mathbb{R}_0^+, \lambda_q)$.

(b) $\mathcal{G}(\mathbb{R}_0^+, m) \subsetneq \mathcal{G}(\mathbb{R}_0^+, \lambda_1)$ and $\overline{\lambda}_1(F) = \overline{m}(F)$, for all $F \in \mathcal{G}(\mathbb{R}_0^+, m)$.

In fact, in all cases $0 < q \leq 1$, take any $F \in \mathcal{G}(\mathbb{R}_0^+, m) \subseteq \mathcal{G}(\mathbb{R}_0^+, \lambda_q)$ and $g \in L^1(\mathbb{R}_0^+, \lambda_q)$. By Theorem 3.1*(a)* and Lemma 3.4,

$$\lim_{n\to\infty} \lambda_q((F \circ T^n)g) = \lim_{n\to\infty} m((F \circ T^n) g h_{\lambda_q}) = \overline{m}(F) m(g h_{\lambda_q}) = \overline{\lambda}_q(F) \lambda_q(g). \tag{37}$$

This simultaneously gives assertions *(b)* and *(c)* of Theorem 3.1. Q.E.D.

4 Proof of the main theorem

We have already mentioned that the maps studied in Section 3 are analogous to the maps on the unit interval that are the object of this proof. Our strategy, in fact, will be to reduce the problem to an application of Theorem 3.1, by means of a suitable conjugation. Of the infinitely many isomorphisms between $(0,1]$ and $\mathbb{R}_0^+$, it turns out that one of the most convenient for our purposes is $\Psi : (0,1] \longrightarrow \mathbb{R}_0^+$, defined by

$$\Psi(x) := \int_x^1 h_{\nu_p}(\xi)\, d\xi = \int_x^1 \xi^{-p-1}\, d\xi = \frac{x^{-p} - 1}{p}. \tag{38}$$

Throughout this section we indicate with a subscript $_o$ all objects pertaining to the space $\mathbb{R}_0^+$. Specifically:

1. Given a map $T : (0,1] \longrightarrow (0,1]$, we denote by $T_o := \Psi \circ T \circ \Psi^{-1}$ its conjugate on $\mathbb{R}_0^+$.
2. Given a measure ν on $(0,1]$ that is infinite at 0, its push-forward $\nu_o := \Psi_* \nu = \nu \circ \Psi^{-1}$ is an infinite, locally finite measure on $\mathbb{R}_0^+$. Observe that if ν is Lebesgue-absolutely continuous, then so is ν_o.
3. For any local observable $f \in L^1((0,1], \nu)$ it follows from the definition of ν_o that $f_o := f \circ \Psi^{-1} \in L^1(\mathbb{R}_0^+, \nu_o)$.
4. For any global observable $F \in \mathcal{G}((0,1], \nu)$ it is readily verified that the corresponding observable $F_o := F \circ \Psi^{-1}$ belongs to $\mathcal{G}(\mathbb{R}_0^+, \nu_o)$, and $\overline{\nu}(F) = \overline{\nu}_o(F_o)$.

The map $f \mapsto f_o$ defined in points 3 and 4 for local and global observables, respectively, is clearly a bijection. Let us call $\mathcal{L}_o$ and $\mathcal{G}_o$, respectively, the images of $\mathcal{L}$ and $\mathcal{G}$. This shows that the isomorphism of non-singular dynamical systems $((0,1], \nu, T) \cong (\mathbb{R}_0^+, \nu_o, T_o)$ also preserves the infinite-volume structure and the property of global-local mixing. More precisely, the former system is global-local mixing relative to $\nu, \mathcal{G}, \mathcal{L}$ if and only if the latter is global-local mixing relative to $\nu_o, \mathcal{G}_o, \mathcal{L}_o$. Hence, the former is fully global-local mixing relative to ν if and only if the latter is fully global-local mixing relative to ν_o.

Let us therefore prove that, for any T satisfying the hypotheses of Theorem 2.4, the conjugate T_o verifies (B1)-(B3). (B1) is immediate since Ψ is a diffeomorphism. (B2) comes from Theorem 2.1. As for (B3), we observe that the inverse branches of T_o are related to those of T by the same conjugation, i.e., $\phi_{o,k} = \Psi \circ \phi_k \circ \Psi^{-1}$, whence

$$\begin{aligned} \phi'_{o,k}(y) &= \Psi'\big(\phi_k(\Psi^{-1}(y))\big)\ \phi'_k\big(\Psi^{-1}(y)\big)\ \frac{1}{\Psi'(\Psi^{-1}(y))} = \\ &= \left(\frac{\phi_k(\Psi^{-1}(y))}{\Psi^{-1}(y)} \right)^{-p-1} \phi'_k\big(\Psi^{-1}(y)\big) \, . \end{aligned} \tag{39}$$

Here we have used that $\Psi'(x) = -x^{-p-1}$. Summing over $k \geq j$ and composing with $y = \Psi(\xi)$, which is a decreasing function of ξ, proves that (B3) and (A5) are equivalent statements. Thus Theorem 3.1 applies. This implies that T is fully global-local mixing relative to both $\Psi_*^{-1} m_o$ and $\Psi_*^{-1} \lambda_q$, for all $q \in (0,1)$, and it is global-local mixing w.r.t. $\Psi_*^{-1} \lambda_1$, $\mathcal{G}((0,1], \Psi_*^{-1} m_o)$, $L^1((0,1], \Psi_*^{-1} \lambda_1)$.

Now, recalling the definition (5) of ν_p, a straightforward calculation based on (38) shows that, for any Lebesgue-absolutely continuous measure ν_o on $\mathbb{R}_0^+$,

$$\frac{d(\Psi_*^{-1} \nu_o)}{d\nu_p} = \frac{d\nu_o}{dm_o} \circ \Psi, \tag{40}$$

where m_o is the Lebesgue measure on $\mathbb{R}_0^+$. This implies in particular that

$$\Psi_*^{-1} m_o = \nu_p. \tag{41}$$

Also, in light of (26), for all $q \in (0,1]$ and $x \in (0,1]$ we have

$$\frac{d(\Psi_*^{-1}\lambda_q)}{d\nu_p}(x) = \frac{1}{(1+\Psi(x))^q}. \tag{42}$$

As $x \to 0^+$, the above expression is asymptotic to $(px^p)^q$, cf. (38). Thus, for $x \to 0^+$,

$$\frac{d(\Psi_*^{-1}\lambda_q)}{dm}(x) \sim p^q\, x^{pq-p-1}, \tag{43}$$

whence, by Theorem 2.1*(a)*,

$$\frac{d(\Psi_*^{-1}\lambda_{1/p})}{d\mu}(x) \sim \frac{p^{1/p}}{H_\mu(0)}. \tag{44}$$

Remark 2 then shows that $\overline{\Psi_*^{-1}\lambda_{1/p}} = \overline{\mu}$, in the sense of Definition 2.3. This fact and (41) show that statements *(a), (b), (c)* of Theorem 2.4 come from the corresponding statements of Theorem 3.1, with $q = 1/p$. Q.E.D.

A Appendix: Technical results

A.1 Proof of Lemma 3.3

We start by recalling a famous result by Lin [8]: T is exact if and only if $\|P^n f\|_1 \to 0$, as $n \to \infty$, for all $f \in L^1$ with $\nu(f) = 0$. Thus, for all $F \in L^\infty$ and $g \in L^1$ with $\nu(g) = 0$,

$$\lim_{n\to\infty} |\nu((F \circ T^n)g)| = \lim_{n\to\infty} |\nu(F\, P^n g)| \le \|F\|_\infty \lim_{n\to\infty} \|P^n g\|_1 = 0. \tag{45}$$

The property that $\nu((F \circ T^n)g)$ vanishes, as $n \to \infty$, for all $F \in \mathcal{G}$ and $g \in \mathcal{L}$ is called **(GLM1)** (w.r.t. $\nu, \mathcal{G}, \mathcal{L}$); cf. [6, 7, 2].

Lemma A.1 *Under the hypotheses of Lemma 3.3, consider $F \in L^\infty$. If, for some $\ell \in \mathbb{R}$ and $\varepsilon \ge 0$, the limit*

$$\limsup_{n\to\infty} \left| \frac{\nu((F \circ T^n)g)}{\nu(g)} - \ell \right| \le \varepsilon$$

holds for some $g \in L^1$ (with $\nu(g) \neq 0$), then it holds for all $g \in L^1$ (with $\nu(g) \neq 0$).

Proof of Lemma A.1. Suppose the above limit holds for $g_0 \in L^1$. Take any other $g \in \mathcal{L}$, with $\nu(g) \neq 0$. We have:

$$\begin{aligned} &\left| \frac{\nu((F \circ T^n)g)}{\nu(g)} - \ell \right| \\ &\qquad \le \left| \nu\left((F \circ T^n) \left(\frac{g}{\nu(g)} - \frac{g_0}{\nu(g_0)} \right) \right) \right| + \left| \frac{\nu((F \circ T^n)g_0)}{\nu(g_0)} - \ell \right|. \end{aligned} \tag{46}$$

The first term of the above r.h.s. vanishes, as $n \to \infty$, due to (45). Q.E.D.

At this point assertion *(a)* of Lemma 3.3 is all but proved. For any $g \in L^1$ with $\nu(g) \neq 0$ one applies the above lemma with $\ell := \overline{F}$ and $\varepsilon := 0$. In the case $\nu(g) = 0$, one applies (45) directly.

As for *(b)*, observe that, since $(\mathcal{M}, \mathscr{A}, \nu)$ is σ-finite and infinite, one can always find a set $A \in \mathscr{A}$ whose measure is finite but larger than any predetermined number. Thus, for any $\varepsilon > 0$, there exists $A \in \mathscr{A}$ with $\|f\|_1/\varepsilon < \nu(A) < \infty$. Set $g_\varepsilon := 1_A/\nu(A)$. We have that

$$\left| \frac{\nu((f \circ T^n) g_\varepsilon)}{\nu(g_\varepsilon)} \right| = |\nu((f \circ T^n) g_\varepsilon)| \leq \|f\|_1 \, \|g_\varepsilon\|_\infty \leq \varepsilon. \tag{47}$$

By Lemma A.1,

$$\limsup_{n \to \infty} \left| \frac{\nu((f \circ T^n) g)}{\nu(g)} \right| \leq \varepsilon \tag{48}$$

holds for all $g \in L^1$ with $\nu(g) \neq 0$. Since ε is arbitrary, we get that the above r.h.s. is zero. When $\nu(g) = 0$, the fact that $\lim_{n\to\infty} \nu((f \circ T^n) g) = 0$ follows directly from (45). This ends the proof of Lemma 3.3. Q.E.D.

A.2 Proof of Lemma 3.4

We start with some preliminary results.

Proposition A.2 *For $q \in (0, 1]$, $\mathcal{G}(\mathbb{R}_0^+, m) \subseteq \mathcal{G}(\mathbb{R}_0^+, \lambda_q)$ and $\overline{\lambda}_q(F) = \overline{m}(F)$, for all $F \in \mathcal{G}(\mathbb{R}_0^+, m)$.*

PROOF. Let $F \in \mathcal{G}(\mathbb{R}_0^+, m)$. Without loss of generality we suppose $\overline{m}(F) = 0$. Therefore, it is enough to show that

$$\lim_{a \to \infty} \frac{1}{\lambda_q([0, a])} \int_0^a F \, d\lambda_q = 0. \tag{49}$$

Using integration by parts we have

$$\begin{aligned} \int_0^a F \, d\lambda_q &= \int_0^a F(y) \, \frac{1}{(1+y)^q} \, dy \\ &= (1+a)^{-q} \int_0^a F(y) \, dy + q \int_0^a \frac{1}{(1+y)^{q+1}} \Big(\int_0^y F(s) \, ds \Big) \, dy. \end{aligned} \tag{50}$$

Now fix any $\varepsilon > 0$. By definition of $\overline{m}$, there exists $M = M(\varepsilon) \in \mathbb{R}^+$ such that, for all $a > M$,

$$\left| \int_0^a F(y) \, dy \right| < \varepsilon a. \tag{51}$$

For all $a > M$ we write

$$\begin{aligned}\frac{1}{\lambda_q([0,a])} \int_0^a F \, d\lambda_q = \frac{(1+a)^{-q}}{\lambda_q([0,a])} \int_0^a F(y) \, dy \\ + \frac{q}{\lambda_q([0,a])} \int_0^M \frac{\left(\int_0^y F(s) \, ds\right)}{(1+y)^{q+1}} \, dy \\ + \frac{q}{\lambda_q([0,a])} \int_M^a \frac{\left(\int_0^y F(s) \, ds\right)}{(1+y)^{q+1}} \, dy.\end{aligned} \tag{52}$$

Let us now specialize to the case $q \in (0,1)$. By (51) and using that

$$\lambda_q([0,a]) = \frac{(1+a)^{1-q} - 1}{1-q}, \tag{53}$$

we get

$$\begin{aligned}\left| \frac{1}{\lambda_q([0,a])} \int_0^a F \, d\lambda_q \right| &\le \varepsilon \, \frac{a(1+a)^{-q}}{\lambda_q([0,a])} \\ &\quad + \frac{q}{\lambda_q([0,a])} \left| \int_0^M \frac{\left(\int_0^y F(s) \, ds\right)}{(1+y)^{q+1}} \, dy \right| \\ &\quad + \frac{q}{\lambda_q([0,a])} \int_M^a \frac{\varepsilon y}{(1+y)^{q+1}} \, dy \\ &\le \varepsilon \, \frac{(1-q)a}{1+a-(1+a)^q} \\ &\quad + \frac{q}{\lambda_q([0,a])} \left| \int_0^M \frac{\left(\int_0^y F(s) \, ds\right)}{(1+y)^{q+1}} \, dy \right| \\ &\quad + \varepsilon \, \frac{q}{\lambda_q([0,a])} \int_M^a \frac{1}{(1+y)^q} \, dy \\ &= \varepsilon(1-q) \, \frac{a}{a+o(a)} + \frac{c(F,M,q)}{(1+a)^{1-q}-1} + \varepsilon q \, \frac{\lambda_q([M,a])}{\lambda_q([0,a])},\end{aligned} \tag{54}$$

as $a \to \infty$. Here $c(F,M,q)$ is a constant that depends on F, M and q, but not on a. It follows that

$$\limsup_{a\to\infty} \left| \frac{1}{\lambda_q([0,a])} \int_0^a F \, d\lambda_q \right| \le \varepsilon(1-q) + \varepsilon q = \varepsilon, \tag{55}$$

for all $\varepsilon > 0$, proving (49).

In the case $q = 1$, we use that $\lambda_1([0,a]) = \log(1+a)$. From (52) we derive

$$\begin{aligned}\left| \frac{1}{\lambda_1([0,a])} \int_0^a F \, d\lambda_1 \right| &\le \varepsilon \, \frac{a}{(1+a)\log(1+a)} \\ &\quad + \frac{c(F,M)}{\log(1+a)} \\ &\quad + \varepsilon \, \frac{\log(1+a) - \log(1+M)}{\log(1+a)},\end{aligned} \tag{56}$$

with the obvious meaning of $c(F, M)$. The limit (49) with $q = 1$ follows. Q.E.D.

To complete the proof of Lemma 3.4*(a)* we need the following

Lemma A.3 *The following statements hold true:*

(i) *For* $q \in (0, \frac{1}{2})$, $\mathcal{G}(\mathbb{R}_0^+, \lambda_q) \subseteq \mathcal{G}(\mathbb{R}_0^+, m)$.

(ii) *For any fixed* $q_1 \in (0,1)$, $\mathcal{G}(\mathbb{R}_0^+, \lambda_{q_2}) \subseteq \mathcal{G}(\mathbb{R}_0^+, \lambda_{q_1})$ *for all* $q_2 \in (q_1, \frac{1+q_1}{2})$.

PROOF. We first remark that *(i)* can be thought of as a particular case of *(ii)* with $q_1 = 0$. For the sake of clarity we prefer to state the two results separately.

Let us first prove *(i)*. We show that $F \notin \mathcal{G}(\mathbb{R}_0^+, m)$ implies $F \notin \mathcal{G}(\mathbb{R}_0^+, \lambda_q)$. For $F \in L^\infty(\mathbb{R}_0^+)$, the statement $F \notin \mathcal{G}(\mathbb{R}_0^+, m)$ is equivalent to saying that

$$A := \limsup_{a\to\infty} \frac{1}{a} \int_0^a F(y)\, dy > B := \liminf_{a\to\infty} \frac{1}{a} \int_0^a F(y)\, dy. \tag{57}$$

Thus we can find two increasing, diverging sequences $(\sigma_k)_{k\in\mathbb{N}}$ and $(\tau_k)_{k\in\mathbb{N}}$ in $\mathbb{R}_0^+$ such that

$$\lim_{k\to\infty} \frac{1}{\sigma_k} \int_0^{\sigma_k} F(y)\, dy = A, \qquad \lim_{k\to\infty} \frac{1}{\tau_k} \int_0^{\tau_k} F(y)\, dy = B. \tag{58}$$

Using (50) with $a = \sigma_k$ we obtain

$$\begin{aligned} &\frac{1}{\lambda_q([0,\sigma_k])} \int_0^{\sigma_k} F(y)\, \frac{1}{(1+y)^q}\, dy \\ &\quad = \frac{(1-q)(1+\sigma_k)^{-q}\sigma_k}{(1+\sigma_k)^{1-q}-1}\, \frac{1}{\sigma_k} \int_0^{\sigma_k} F(y)\, dy \\ &\quad\quad + \frac{q}{\lambda_q([0,\sigma_k])} \int_0^{\sigma_k} \frac{1}{(1+y)^{q+1}} \Big(\int_0^y F(s)\, ds \Big) dy. \end{aligned} \tag{59}$$

Now fix $\varepsilon > 0$. By the first limit in (58), there exists $\bar{k} \in \mathbb{N}$ such that, for all $k \geq \bar{k}$,

$$\frac{1}{\sigma_k} \int_0^{\sigma_k} F(y)\, dy > A - \varepsilon. \tag{60}$$

Also, by the definition of B in (57), there exists $\bar{y} \in \mathbb{R}^+$ such that, for all $y > \bar{y}$,

$$\frac{1}{y} \int_0^y F(s)\, ds > B - \varepsilon. \tag{61}$$

Since (σ_k) is diverging, we find $K \geq \bar{k}$ such that $\sigma_k > \bar{y}$, for all $k \geq K$. Therefore, by (59),

$$\begin{aligned} &\frac{1}{\lambda_q([0,\sigma_k])} \int_0^{\sigma_k} \frac{F(y)}{(1+y)^q}\, dy \\ &\quad > \frac{(1-q)(1+\sigma_k)^{-q}\sigma_k}{(1+\sigma_k)^{1-q}-1}\, (A-\varepsilon) + \frac{q}{\lambda_q([0,\sigma_k])} \int_0^{\sigma_k} \frac{(B-\varepsilon)y}{(1+y)^{q+1}}\, dy, \end{aligned} \tag{62}$$

for all $k \geq K$. Rewriting $(B-\varepsilon)y = (B-\varepsilon)((1+y)-1)$, we see that

$$\begin{aligned} \frac{q}{\lambda_q([0,\sigma_k])} \int_0^{\sigma_k} \frac{(B-\varepsilon)y}{(1+y)^{q+1}}\, dy \\ = q(B-\varepsilon)\, \frac{\lambda_q([0,\sigma_k]) - \int_0^{\sigma_k} (1+y)^{-q-1}\, dy}{\lambda_q([0,\sigma_k])} \end{aligned} \tag{63}$$

which, together with (62), leads to

$$\liminf_{k\to\infty} \frac{1}{\lambda_q([0,\sigma_k])} \int_0^{\sigma_k} \frac{F(y)}{(1+y)^q}\, dy > A(1-q) + Bq - \varepsilon. \tag{64}$$

Since the above l.h.s. does not depend on ε, we conclude that

$$\liminf_{k\to\infty} \frac{1}{\lambda_q([0,\sigma_k])} \int_0^{\sigma_k} \frac{F(y)}{(1+y)^q}\, dy \geq A(1-q) + Bq. \tag{65}$$

Analogously, putting $a = \tau_k$ in (50) and writing estimates from the above that are specular to the ones used above, we obtain

$$\limsup_{k\to\infty} \frac{1}{\lambda_q([0,\tau_k])} \int_0^{\tau_k} \frac{F(y)}{(1+y)^q}\, dy \leq B(1-q) + Aq. \tag{66}$$

Since $A > B$ and $q < \frac{1}{2}$, we have that $A(1-q) + Bq > B(1-q) + Aq$. We have therefore produced two subsequences of $a \mapsto \lambda_q([0,a])^{-1} \int_0^a F\, d\lambda_q$ with different limits. This shows that $F \notin \mathcal{G}(\mathbb{R}_0^+, \lambda_q)$ and the proof of *(i)* is finished.

We now show *(ii)* using a similar reasoning. Let $F \in L^\infty(\mathbb{R}_0^+)$ with $F \notin \mathcal{G}(\mathbb{R}_0^+, \lambda_{q_1})$. This means that

$$A := \limsup_{a\to\infty} \frac{1}{\lambda_{q_1}([0,a])} \int_0^a F\, d\lambda_{q_1} > B := \liminf_{a\to\infty} \frac{1}{\lambda_{q_1}([0,a])} \int_0^a F\, d\lambda_{q_1}, \tag{67}$$

implying the existence of two increasing sequences $(\sigma_k)_{k\in\mathbb{N}}, (\tau_k)_{k\in\mathbb{N}} \subset \mathbb{R}_0^+$ such that $\sigma_k, \tau_k \to \infty$ and

$$\lim_{k\to\infty} \frac{1}{\lambda_{q_1}([0,\sigma_k])} \int_0^{\sigma_k} F\, d\lambda_{q_1} = A, \qquad \lim_{k\to\infty} \frac{1}{\lambda_{q_1}([0,\tau_k])} \int_0^{\tau_k} F\, d\lambda_{q_1} = B. \tag{68}$$

We need an adapted version of (50). Integrating by parts we have

$$\begin{aligned} \int_0^a F\, d\lambda_{q_2} &= \int_0^a F(y)\, \frac{1}{(1+y)^{q_2}}\, dy \\ &= (1+a)^{q_1-q_2} \int_0^a \frac{F(y)}{(1+y)^{q_1}}\, dy \\ &\quad + (q_2-q_1) \int_0^a \frac{1}{(1+y)^{q_2-q_1+1}} \left(\int_0^y \frac{F(s)}{(1+s)^{q_1}}\, ds \right) dy, \end{aligned} \tag{69}$$

where we recall that $q_2 > q_1$. By means of (53), we write

$$\begin{aligned}
&\frac{1}{\lambda_{q_2}([0,a])} \int_0^a F \, d\lambda_{q_2} = \\
&\qquad = \frac{1-q_2}{1-q_1} \, \frac{(1+a)^{1-q_2} - (1+a)^{q_1-q_2}}{(1+a)^{1-q_2} - 1} \, \frac{1}{\lambda_{q_1}([0,a])} \int_0^a F(y) \, d\lambda_{q_1}(y) \\
&\qquad\quad + \frac{q_2 - q_1}{\lambda_{q_2}([0,a])} \int_0^a \frac{1}{(1+y)^{q_2-q_1+1}} \left(\int_0^y \frac{F(s)}{(1+s)^{q_1}} \, ds \right) dy.
\end{aligned} \tag{70}$$

Now set $a = \sigma_k$ in (70). Starting from (67) one can repeat the same arguments that were used to show (62) to prove that, for all $\varepsilon > 0$,

$$\begin{aligned}
&\liminf_{k\to\infty} \frac{1}{\lambda_{q_2}([0,\sigma_k])} \int_0^{\sigma_k} F(y) \, d\lambda_{q_2}(y) \\
&\qquad > (A - \varepsilon) \, \frac{1-q_2}{1-q_1} + \liminf_{k\to\infty} \frac{q_2 - q_1}{\lambda_{q_2}([0,\sigma_k])} \int_0^{\sigma_k} \frac{(B-\varepsilon) \, \lambda_{q_1}([0,y])}{(1+y)^{q_2-q_1+1}} \, dy.
\end{aligned} \tag{71}$$

In analogy with (63), and using (53), we conclude that

$$\begin{aligned}
&\liminf_{k\to\infty} \frac{1}{\lambda_{q_2}([0,\sigma_k])} \int_0^{\sigma_k} \frac{(B-\varepsilon) \, \lambda_{q_1}([0,y])}{(1+y)^{q_2-q_1+1}} \, dy \\
&\qquad = (B-\varepsilon) \, \liminf_{k\to\infty} \frac{\int_0^{\sigma_k} (1+y)^{-q_2} \, dy - \int_0^{\sigma_k} (1+y)^{-q_2+q_1-1} \, dy}{(1-q_1) \, \lambda_{q_2}([0,\sigma_k])} \\
&\qquad = \frac{B-\varepsilon}{1-q_1}.
\end{aligned} \tag{72}$$

Once again, using that the l.h.s. of (71) does not depend on ε, we obtain

$$\liminf_{k\to\infty} \frac{1}{\lambda_{q_2}([0,\sigma_k])} \int_0^{\sigma_k} F(y) \, d\lambda_{q_2}(y) \ge \frac{A(1-q_2) + B(q_2-q_1)}{1-q_1}. \tag{73}$$

Just as before, one can set $a = \tau_k$ in (70) and take the limsup of the two sides to find

$$\limsup_{k\to\infty} \frac{1}{\lambda_{q_2}([0,\tau_k])} \int_0^{\tau_k} F(y) \, d\lambda_{q_2}(y) \le \frac{B(1-q_2) + A(q_2-q_1)}{1-q_1}. \tag{74}$$

On the other hand, for $q_1 \in (0,1)$ and $q_2 \in (q_1, \frac{1+q_1}{2})$, the inequality

$$\frac{A(1-q_2) + B(q_2-q_1)}{1-q_1} > \frac{B(1-q_2) + A(q_2-q_1)}{1-q_1} \tag{75}$$

is equivalent to $A > B$. So, in analogy to what was done earlier, we have found two subsequences of $a \mapsto \lambda_{q_2}([0,a])^{-1} \int_0^a F \, d\lambda_{q_2}$ with different limits. Then $F \notin \mathcal{G}(\mathbb{R}_0^+, \lambda_{q_2})$, ending the proof of *(ii)*. Q.E.D.

PROOF OF LEMMA 3.4*(a)*. Let $q \in (0,1)$. The inclusion $\mathcal{G}(\mathbb{R}_0^+, m) \subseteq \mathcal{G}(\mathbb{R}_0^+, \lambda_q)$ and the equality $\overline{\lambda}_q(F) = \overline{m}(F)$, for all $F \in \mathcal{G}(\mathbb{R}_0^+, m)$, were proved in Proposition A.2.

As for the opposite inclusion, it was proved in Lemma A.3*(i)*, for the case $q \in (0, \frac{1}{2})$. When $q = \frac{1}{2}$, using Lemma A.3*(ii)* with $q_1 := \frac{1}{4}$ and $q_2 := q = \frac{1}{2} \in (\frac{1}{4}, \frac{5}{8})$, we get

$$\mathcal{G}(\mathbb{R}_0^+, \lambda_{1/2}) \subseteq \mathcal{G}(\mathbb{R}_0^+, \lambda_{1/4}) \subseteq \mathcal{G}(\mathbb{R}_0^+, m). \tag{76}$$

Suppose instead that $q \in (\frac{1}{2}, \frac{3}{4})$. By Lemma A.3*(ii)* with $q_1 := \frac{1}{2}$ and $q_2 := q \in (q_1, \frac{1+q_1}{2})$, we have

$$\mathcal{G}(\mathbb{R}_0^+, \lambda_q) \subseteq \mathcal{G}(\mathbb{R}_0^+, \lambda_{1/2}) \subseteq \mathcal{G}(\mathbb{R}_0^+, m). \tag{77}$$

We can now iterate the argument to obtain $\mathcal{G}(\mathbb{R}_0^+, \lambda_q) \subseteq \mathcal{G}(\mathbb{R}_0^+, m)$, for all $q \in [1 - 2^{-n}, 1 - 2^{-n-1})$, for all $n \geq 1$. Assertion *(a)* is proved. Q.E.D.

PROOF OF LEMMA 3.4*(b)*. The facts that $\mathcal{G}(\mathbb{R}_0^+, m) \subseteq \mathcal{G}(\mathbb{R}_0^+, \lambda_1)$ and $\overline{\lambda}_1(F) = \overline{m}(F)$, for all $F \in \mathcal{G}(\mathbb{R}_0^+, m)$, are proved in Proposition A.2. It remains to show that the inclusion is actually strict, that is, there exists $F \in \mathcal{G}(\mathbb{R}_0^+, \lambda_1) \setminus \mathcal{G}(\mathbb{R}_0^+, m)$.

For $k \in \mathbb{N}$, set

$$\alpha_k := k^k - 1, \qquad \beta_k := 2k^k - 1. \tag{78}$$

Notice that $\alpha_1 = 0$, $\alpha_k < \beta_k < \alpha_{k+1}$ for all k, and $\alpha_k, \beta_k \to \infty$, as $k \to \infty$. Now define a function $F : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ as follows. For all $k \in \mathbb{N}$,

$$F(y) := \begin{cases} 1, & y \in [\alpha_k, \beta_k); \\ 0, & y \in [\beta_k, \alpha_{k+1}), \end{cases} \tag{79}$$

We first show that $F \in \mathcal{G}(\mathbb{R}_0^+, \lambda_1)$ and in particular that

$$\overline{\lambda}_1(F) = \lim_{a \to \infty} \frac{1}{\lambda_1([0,a])} \int_0^a F(y) \, \frac{1}{1+y} \, dy = 0. \tag{80}$$

Letting $a = \alpha_n$, for $n > 1$, we find

$$\begin{aligned} \frac{1}{\lambda_1([0,\alpha_n])} \int_0^{\alpha_n} F(y) \, \frac{1}{1+y} \, dy &= \frac{1}{\log(1+\alpha_n)} \sum_{k=1}^{n-1} \int_{\alpha_k}^{\beta_k} \frac{1}{1+y} \, dy \\ &= \frac{1}{\log(1+\alpha_n)} \sum_{k=1}^{n-1} \log\left(\frac{1+\beta_k}{1+\alpha_k}\right) \\ &= \frac{1}{n \log n} \sum_{k=1}^{n-1} \log 2 = O\left(\frac{1}{\log n}\right), \end{aligned} \tag{81}$$

as $n \to \infty$. In the same way, for $a = \beta_n$, we find

$$\begin{aligned}\frac{1}{\lambda_1([0,\beta_n])} \int_0^{\beta_n} F(y) \frac{1}{1+y}\, dy &= \frac{1}{\log(1+\beta_n)} \sum_{k=1}^{n} \int_{\alpha_k}^{\beta_k} \frac{1}{1+y}\, dy \\ &= \frac{1}{\log(1+\beta_n)} \sum_{k=1}^{n} \log\left(\frac{1+\beta_k}{1+\alpha_k}\right) \\ &= \frac{1}{\log 2 + n\log n} \sum_{k=1}^{n} \log 2 = O\!\left(\frac{1}{\log n}\right).\end{aligned} \tag{82}$$

Moreover,

$$\forall a \in (\alpha_n, \beta_n), \quad \frac{1}{\lambda_1([0,a])} \int_0^a F(y)\, d\lambda_1(y) \le \frac{1}{\lambda_1([0,\alpha_n])} \int_0^{\beta_n} F(y)\, d\lambda_1(y); \tag{83}$$

$$\forall a \in (\beta_n, \alpha_{n+1}), \quad \frac{1}{\lambda_1([0,a])} \int_0^a F(y)\, d\lambda_1(y) \le \frac{1}{\lambda_1([0,\beta_n])} \int_0^{\alpha_{n+1}} F(y)\, d\lambda_1(y). \tag{84}$$

Since $\lambda_1([0,\alpha_n]) \sim \lambda_1([0,\alpha_{n+1}]) \sim \lambda_1([0,\beta_n])$, the above l.h.sides vanish, as $n \to \infty$. Hence (80) is proved and $F \in \mathcal{G}(\mathbb{R}_0^+, \lambda_1)$.

Lastly, we prove that $F \notin \mathcal{G}(\mathbb{R}_0^+, m)$ by showing that the infinite-volume average

$$\overline{m}(F) = \lim_{a\to\infty} \frac{1}{a} \int_0^a F(y)\, dy \tag{85}$$

does not exist. In fact, let $a = \alpha_n$, with $n > 1$. Then

$$\begin{aligned}\frac{1}{\alpha_n} \int_0^{\alpha_n} F(y)\, dy &= \frac{1}{\alpha_n} \sum_{k=1}^{n-1} \int_{\alpha_k}^{\beta_k} dy = \frac{1}{\alpha_n} \sum_{k=1}^{n-1} (\beta_k - \alpha_k) \\ &= \frac{1}{n^n - 1} \sum_{k=1}^{n-1} k^k \le \frac{1}{n^n - 1} \sum_{k=1}^{n-1} n^k = O\!\left(\frac{1}{n}\right),\end{aligned} \tag{86}$$

as $n \to \infty$. On the other hand, for $a = \beta_n$ and $n > 1$, we see that

$$\begin{aligned}\frac{1}{\beta_n} \int_0^{\beta_n} F(y)\, dy &= \frac{1}{\beta_n} \sum_{k=1}^{n} \int_{\alpha_k}^{\beta_k} dy = \frac{1}{\beta_n} \sum_{k=1}^{n} (\beta_k - \alpha_k) \\ &= \frac{1}{2n^n - 1} \sum_{k=1}^{n} k^k = \frac{n^n}{2n^n - 1} + \frac{1}{2n^n - 1} \sum_{k=1}^{n-1} k^k \ge \frac{1}{2}.\end{aligned} \tag{87}$$

Therefore $F \notin \mathcal{G}(\mathbb{R}_0^+, m)$ and the lemma is proved. Q.E.D.

References

[1] C. Bonanno, P. Giulietti and M. Lenci, *Global-local mixing for the Boole map*, Chaos Solitons Fractals **111** (2018), 55–61.

[2] C. Bonanno, P. Giulietti and M. Lenci, *Infinite mixing for one-dimensional maps with an indifferent fixed point*, Nonlinearity **31** (2018), no. 11, 5180–5213.

[3] D. Dolgopyat and P. Nándori, *Infinite measure mixing for some mechanical systems*, `arXiv:1812.01174`.

[4] P. Giulietti, A. Hammerlindl and D. Ravotti, *Quantitative global-local mixing mixing for accessible skew products*, `arXiv:2006.06539`.

[5] M. Lenci, *On infinite-volume mixing*, Comm. Math. Phys. **298** (2010), no. 2, 485–514.

[6] M. Lenci, *Exactness, K-property and infinite mixing*, Publ. Mat. Urug. **14** (2013), 159–170.

[7] M. Lenci, *Uniformly expanding Markov maps of the real line: exactness and infinite mixing*, Discrete Contin. Dyn. Syst. **37** (2017), no. 7, 3867–3903.

[8] M. Lin, *Mixing for Markov operators*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **19** (1971), 231–242.

[9] C. Liverani, B. Saussol and S. Vaienti, *A probabilistic approach to intermittency*, Ergodic Theory Dynam. Systems **19** (1999), no. 3, 671–685.

[10] P. Manneville, *Intermittency, self-similarity and 1/f spectrum in dissipative dynamical systems*, J. Physique **41** (1980), no. 11, 1235–1243.

[11] Y. Pomeau and P. Manneville, *Intermittent transition to turbulence in dissipative dynamical systems*, Comm. Math. Phys. **74** (1980), no. 2, 189–197.

[12] M. Thaler, *Estimates of the invariant densities of endomorphisms with indifferent fixed points*, Israel J. Math. **37** (1980), 303–314.

[13] M. Thaler, *Transformations on [0,1] with infinite invariant measures*, Israel J. Math. **46** (1983), no. 1-2, 67–96.

[14] M. Thaler, *Infinite ergodic theory. Examples: One-dimensional maps with indifferent fixed points*, Course notes, Marseille, 2001, available at `www.uni-salzburg.at/fileadmin/oracle_file_imports/1543201.PDF`.